MATH 1105 - FALL 2008 - 12-5-08
SECTION 2
SOLUTIONS TO CHALLENGING REVIEW PROBLEMS

The following are intended to be challenging problems that will help you review the material for the final exam. Each problem is intended to test your knowledge of multiple parts of the material as opposed to a single specific section. In this way these problems will be more like problems that could appear on the final exam than problems from your textbook. On the other hand, in my opinion these problems are largely inappropriate for the final exam for any of a number of reasons including but not limited to being too complicated, too difficult to understand, too wordy, and requiring a calculator. I suggest trying these problems after you have studied significantly to help you see how to various topics from the course interconnect.

(1) (a) Two cards are chosen at random from a normal deck of 52 cards. What is the probability that they are both queens?
$\frac{\binom{4}{2}}{\binom{52}{2}} = .00452$

(b) 10000 two card hands are chosen at random from the deck of cards in (a), where the cards are replaced after each hand is chosen. Approximate the probability that more than 58 of these hands have two queens.
This is a binomial distribution with $p = .00452$ and $n = 10000$. So the distribution is approximated by a normal curve with mean $\mu = 10000 \cdot .00452 = 45.2$ and standard deviation $\sigma = \sqrt{10000 \cdot .00452 \cdot (1 - .00452)} = 6.7115$. The probability that more than 58 have 2 queens is the area under this normal curve to the right of $x = 58.5$. Calculating the z-score of 58.5 translates us to the standard normal curve $z = \frac{58.5-45.2}{6.7115} = 1.974$. The area to the left of $z = 1.97$ in the stanard normal curve is .9756, so the area to the right of $z = 1.97$ is $1 - .9756 = .0244$. The probability that more than 58 of these hands have two queens is approximately .0244.

(c) Two cards are chosen at random from a deck of 32 cards made from a normal deck of cards by removing all cards with values that are even numbers. What is the probability that they are both queens?
$\frac{\binom{4}{2}}{\binom{32}{2}} = .0121$

(d) 10000 two card hands are chosen at random from the deck of cards in (c), where the cards are replaced after each hand is chosen. Approximate the probability that more than 58 of these hands have two queens.
This is a binomial distribution with $p = .0121$ and $n = 10000$. So the distribution is approximated by a normal curve with mean $\mu = 10000 \cdot .0121 = 121$ and

standard deviation $\sigma = \sqrt{10000 \cdot .0121 \cdot (1 - .0121)} = 10.932$. The probability that more than 58 have 2 queens is the area under this normal curve to the right of $x = 58.5$. Calculating the z-score of 58.5 translates us to the standard normal curve $z = \frac{58.5-121}{10.932} = -5.717$. The area to the left of $z = -5.717$ in the stanard normal curve is very close to 0, so the area to the right of $z = -5.717$ is approximately 1. The probability that more than 58 of these hands have two queens is approximately 1.

(e) One of the decks in (a) and (c) is chosen at random and 10000 two card hands are chosen at random from that deck, where the cards are replaced after each hand is chosen. If more than 58 of these hands have two queens, what is the probability that they were chosen from the 32 card deck?
Let E_{58} be the event that more than 58 hands have two queens, and let D_{52} and D_{32} be the events that the 52 card deck and 32 card deck are chosen respectively. Then by Bayes' Theorem $P(D_{32}|E_{58}) = \frac{P(D_{32})P(E_{58}|D_{32})}{P(D_{32})P(E_{58}|D_{32})+P(D_{52})P(E_{58}|D_{52})}$. $P(D_{32}) = P(D_{52}) = .5$ and from (b) and (d), $P(E_{58}|D_{52}) = .0244$ and $P(E_{58}|D_{32}) = 1$. So $P(D_{32}|E_{58}) = \frac{.5 \cdot 1}{.5 \cdot 1 + .5 \cdot .0244} = .976$. If more than 58 of these hands have two queens the probability that they were chosen from the 32 card deck is .976.

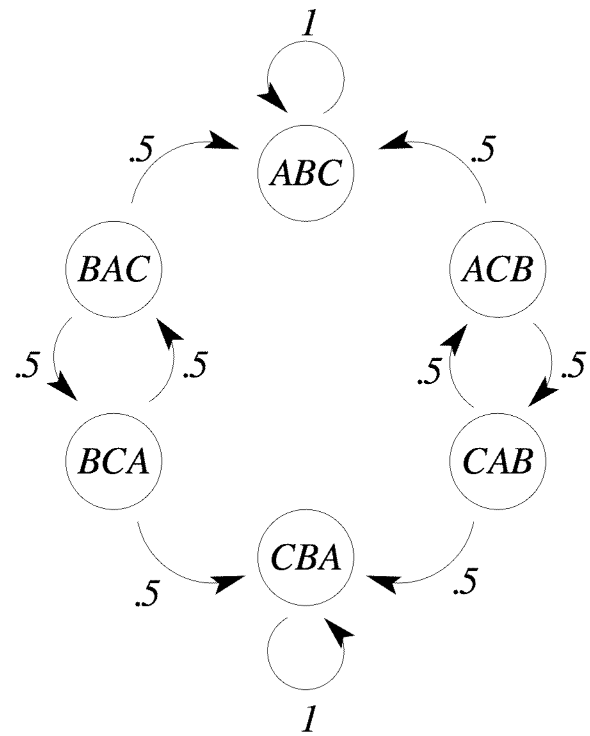
(2) (a) How many different orderings are there for the letters A, B, and C?
$P(3, 3) = 3! = 6$.

(b) What is the probability that a randomly chosen ordering is alphabetical or reverse alphabetical?
There is one alaphabetical ordering A,B,C and one reverse alphabetical ordering C,B,A. The probability is $\frac{2}{6} = \frac{1}{3}$.

(c) Consider the experiment where, given an order on the letters A, B, and C, two subsequent letters are chosen at random and their positions in the order are switched, unless the given order is alphabetical or reverse alphabetical in which case nothing is done. This describes a Markov chain where the states correspond to the orders on the letters A, B, and C. Draw the transition diagram and determine the transition matrix of this Markov chain.

$$\begin{array}{c} ABC \\ CBA \\ ACB \\ CAB \\ BAC \\ BCA \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(d) Is the Markov chain regular?
No, ABC and CBA are absorbing states, and so the first two rows will have five zeroes no matter how many powers of the transition matrix you take.



(e) Is the Markov chain absorbing?
Yes, the Markov chain has two absorbing states and every non-absorbing state can reach an absorbing state.

(f) If the initial state is BAC, what is the probability of ending up in the state CBA?
The fundamental matrix of this absorbing Markov chain is

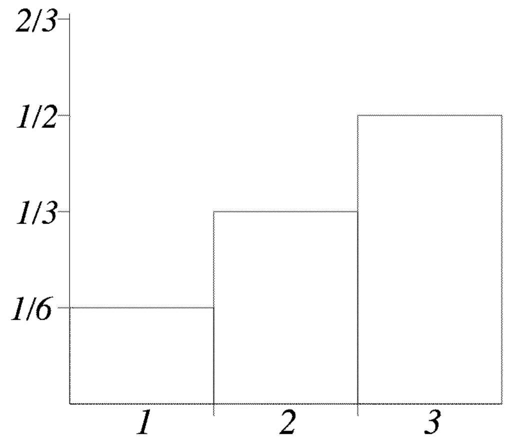
$$F = \begin{array}{c} ACB \\ CAB \\ BAC \\ BCA \end{array} \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{4}{3} \end{bmatrix}.$$

The long range projection for non-absorbing states is

$$FR = \begin{array}{c} ACB \\ CAB \\ BAC \\ BCA \end{array} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

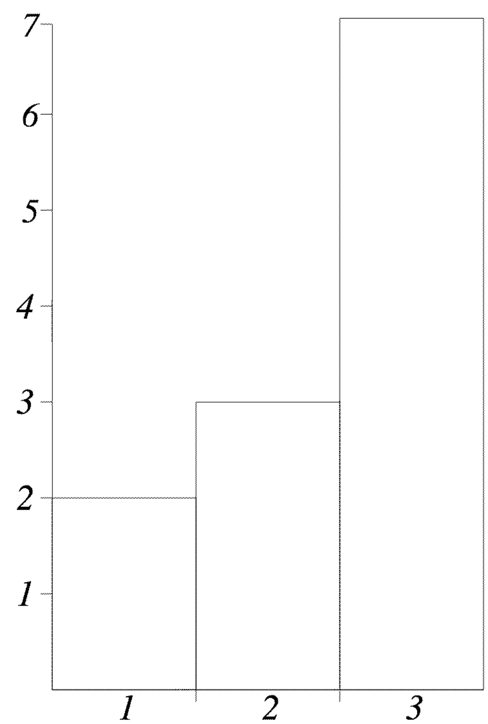
where the first column corresponds to ABC and the second corresponds to CBA. If the initial state is BAC the probability of ending up in the state CBA is $\frac{1}{3}$.

(3) (a) Instead of sides 1 through 6, an unusual die has one side labeled 1, two sides labeled 2, and three sides labeled 3. Draw the histogram for the probability distribution associated to rolling this die.



(b) What is the expected value of rolling this die?
$\frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 2 + \frac{3}{6} \cdot 3 = \frac{7}{3}$.

(c) The die is rolled 12 times yielding the results 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 3, and 3. Draw the histogram given by the frequency distribution determined by this data.



(d) Calculate the mean, median, mode, and standard deviation of this frequency distribution.

Mean $= \frac{1+1+2+2+2+3+3+3+3+3+3+3}{12} = \frac{29}{12}$, Median $= 3$, Mode $= 3$, Standard Deviation $= \sqrt{\frac{1^2+1^2+2^2+2^2+2^2+3^2+3^2+3^2+3^2+3^2+3^2+3^2-12(\frac{29}{12})^2}{11}} = .793$

(e) Let x_1, x_2, and x_3 be the outcomes of this frequency distribution, and $f(x_i)$ be the frequency of the outcome x_i. The data points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ define a least squares line. What is the formula of this least squares line?
Recall that for data points (x_1, y_1), (x_2, y_2), and (x_3, y_3), if the least square line for this data is $y = mx + b$, then $m = \frac{3\Sigma_{i=1}^{3}(x_i y_i)-(\Sigma_{i=1}^{3}(x_i))(\Sigma_{i=1}^{3}(y_i))}{3(\Sigma_{i=1}^{3}(x_i^2))-(\Sigma_{i=1}^{3}(x_i))^2}$, and $b = \frac{(\Sigma_{i=1}^{3}(y_i))-m(\Sigma_{i=1}^{3}(x_i))}{3}$. Plugging in the x_i and y_i yields $m = \frac{3(2+6+21)-6(12)}{3(14)-6^2} = \frac{5}{2}$ and $b = \frac{12-2.5(6)}{3} = -1$. The least squares line for this data is $y = \frac{5}{2}x - 1$.

(4) (a) The marginal cost for cereal companies is normally distributed with mean \$.40 and standard deviation \$.05. A cereal company is chosen at random. What is the probability that it costs between \$.45 and \$.50 for this company to make a box of cereal?
The probability that the marginal cost for the company is between \$.45 and \$.50 is the area under the normal curve with mean \$.40 and standard deviation \$.05 between $x = .5$ and $x = .45$. Translating to the standard normal curve is accomplished by calculating the z-scores of these points, in particular $z_1 = \frac{.5-.4}{.05} = 2$ and $z_2 = \frac{.45-.4}{.05} = 1$. The area between $z_1 = 2$ and $z_2 = 1$ under the standard normal curve is the area to the left of $z = 2$ minus the area to the left of $z = 1 = .9772 - .8413 = .1359$. The probability that it costs between \$.45 and \$.50 for this company to make a box of cereal is .1359.

(b) A particular cereal companies' marginal cost is less than the marginal cost of 75% of cereal companies. What is the marginal cost for this company?
The z value such that the area to the left of z under the standard normal curve is .25 is $-.67$, hence this value has .75 area under the standard normal curve to its right. This z-score corresponds to \$.05($-.67$)+\$.4 =\$.367.

(c) If the company in part (b) has fixed cost \$50,000 and sells each box of cereal for \$3, how many boxes must the company produce and sell in order to break even?
The cost function for this company is $C(x) =$\$50000+\$.367x and the revenue function is $R(x) =$\$3x. Setting $C(x) = R(x)$ yields the break even point, $x =$ 18987 boxes of cereal must be produced and sold for the company to break-even.

(5) (a) Balls labeled 1,2,3, and 4 are in a bag. If two balls are chosen at random, what is the probability that there sum will be an even number?
Let E be the event that the sum of the two balls is even and S be a sample space of equally likely outcomes. Then $P(E) = \frac{n(E)}{n(S)} = \frac{2}{\binom{4}{2}} = \frac{1}{3}$.

(b) Consider the following experiment preformed on a set of two balls that have been removed from the bag. Of the two balls not in the bag, one is chosen and returned to the bag. The ball with the lower number is chosen $\frac{1}{3}$ of the time and the ball with the higher number is chosen $\frac{2}{3}$ of the time. Then one ball is chosen at random from the bag, and this ball together with the ball that remained outside the bag is the next pair of balls. If 1 and 2 were outside of the bag what is the probability that after one trial 2 and 4 are outside of the bag? that 1 and 2 are still outside of the bag?

If 1 and 2 were outside of the bag the probability that after one trial 2 and 4 are outside of the bag = (probabilty that between 1 and 2, 1 is selected) (probabilty the in a bag with three balls, the ball labeled 4 is chosen) $= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. If 1 and 2 were outside of the bag the probability that after one trial 2 and 4 are outside of the bag = (probabilty that between 1 and 2, 1 is selected) (probabilty the in a bag with three balls, the ball labeled 1 is chosen) + (probabilty that between 1 and 2, 2 is selected) (probabilty the in a bag with three balls, the ball labeled 2 is chosen) $= \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{3}$.

(c) The experiment from (b) describes a Markov chain, where the states correspond to the different possible pairs of balls. Draw the transition diagram and determine the transition matrix of this Markov chain.

$$P = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{bmatrix} \frac{1}{3} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{3} & 0 & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{1}{9} & 0 & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & 0 & \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{3} \end{bmatrix}$$

(d) Is this Markov chain absorbing? regular? Explain.

P has no absorbing states so the Markov chain is not absorbing. The Markov chain is regular because P^2 has no zero entries:

$$P^2 = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{bmatrix} \frac{7}{27} & \frac{17}{81} & \frac{17}{81} & \frac{10}{81} & \frac{10}{81} & \frac{2}{27} \\ \frac{2}{9} & \frac{20}{81} & \frac{17}{81} & \frac{10}{81} & \frac{7}{81} & \frac{1}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{19}{81} & \frac{8}{81} & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} & \frac{2}{9} & \frac{5}{27} & \frac{1}{9} \\ \frac{2}{9} & \frac{10}{81} & \frac{11}{81} & \frac{16}{81} & \frac{17}{81} & \frac{1}{9} \\ \frac{4}{27} & \frac{16}{81} & \frac{11}{81} & \frac{16}{81} & \frac{11}{81} & \frac{5}{27} \end{bmatrix}$$

(e) Does this Markov chain have an equilibrium vector? If so, find it. If not, explain why not.

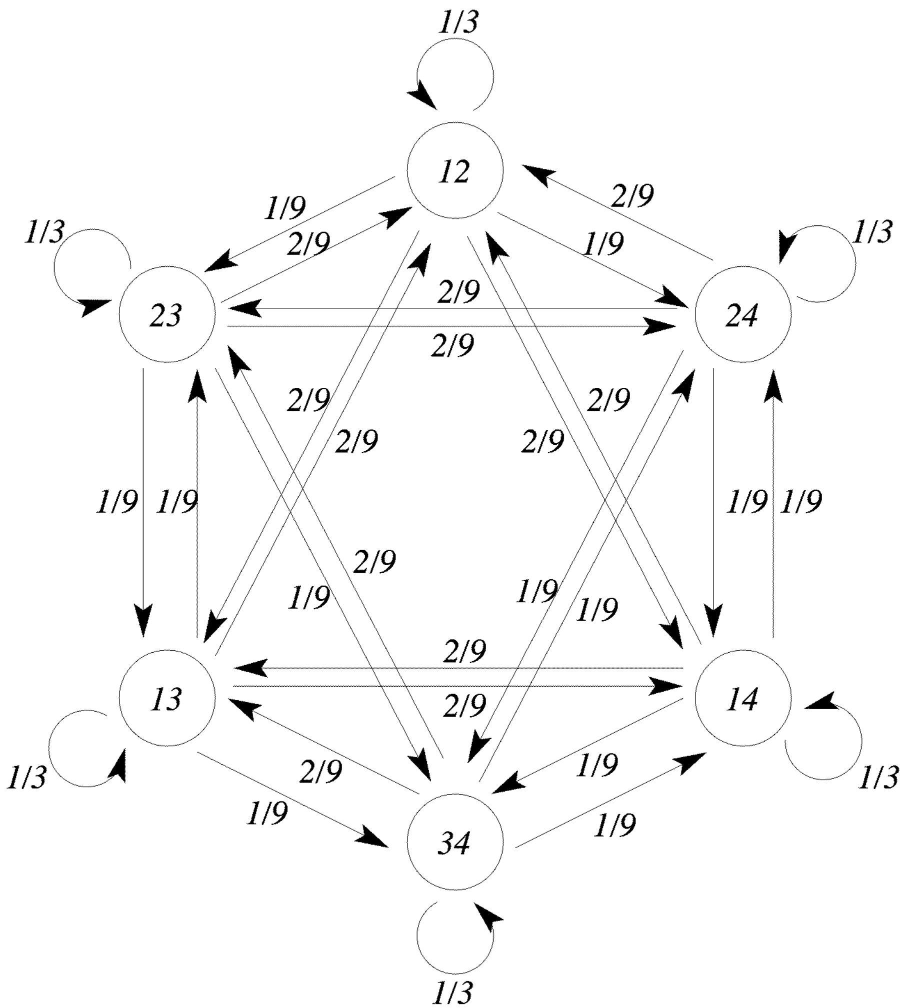
Yes, it does because the Markov chain is regular. It is

$$\begin{bmatrix} \frac{2}{9} & \frac{62}{315} & \frac{19}{105} & \frac{16}{105} & \frac{43}{315} & \frac{1}{9} \end{bmatrix}.$$

(6) (a) A survey of 50 girl scouts is taken to determine which flavors of girl scout cookies they like. The survey yields the following data:

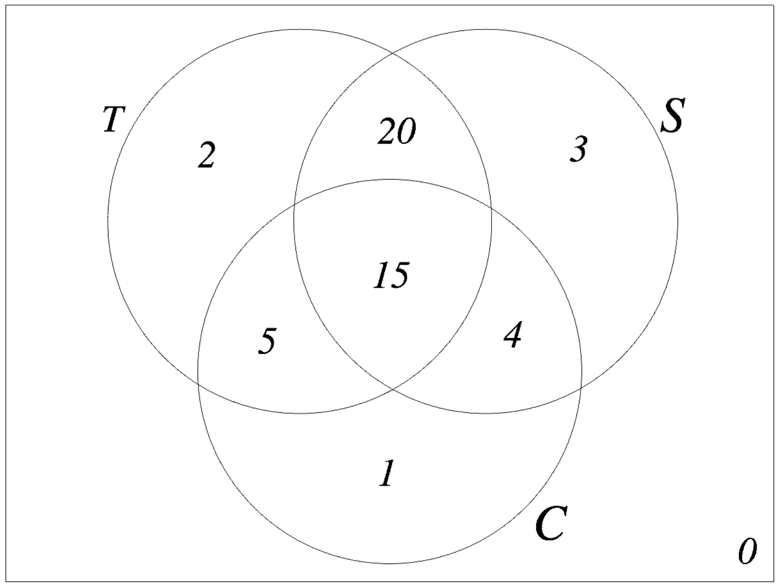
42 girl scouts like Thin Mints

42 girl scouts like Shortbread



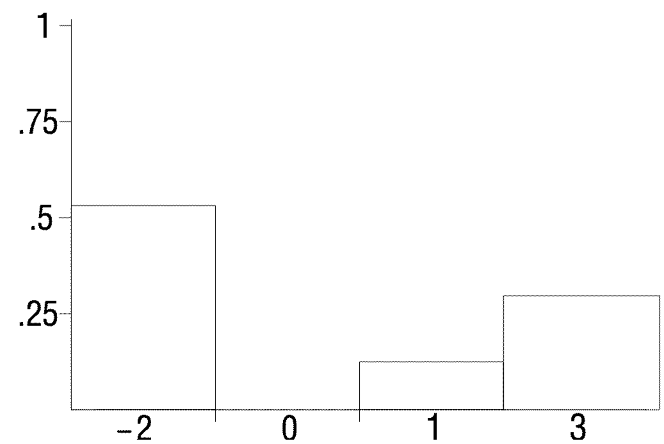
24 girl scouts like Caramel deLites
35 girl scouts like Thin Mints and Shortbread
20 girl scouts like Thin Mints and Caramel deLites
19 girl scouts like Shortbread and Caramel deLites
0 girl scouts do not like Thin Mints, Shortbread, nor Caramel deLites.
How many girl scouts like Thin Mints, Shortbread, and Caramel deLites? Draw and completly fill in a Venn Diagram for this survey data:
15 girl scouts like Thin Mints, Shortbread, and Caramel deLites.

(b) Surveyors Alice and Bob play a game. They select a girl scout at random from the 50 surveyed, and supposed that this girl scout likes exactly y out of the 3 flavors of cookies in the survey. If y is odd Bob pays Alice \$$y$, and if y is even Alice pays Bob \$$y$. Let x be the random variable representing Alice's net profit.



Draw a table and histogram showing the probability distribution of x.

x	$P(x)$
\$3	.3
\$1	.12
\$0	0
$-$\$2	.58



(c) What is the expected value of the probability distribution?
$\$0 \cdot 0 + \$1 \cdot .12 - \$2 \cdot .58 + \$3 \cdot .3 = -\$.14$

(7) (a) A coin is flipped 3 times. What is the sample space of equally likely outcomes for this sample space?
$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$

(b) What is the probability that exactly 2 heads are flipped? Let E be the event that 2 heads are filled. Then $P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$.

(c) Regard flipping a coin 3 times as a trial in an experiment. If 1000 trials are run, what is the exact probability that in 370 of these trials 2 heads will be flipped? This is a binomial distribution with probability $\frac{3}{8}$ and 1000 trials. So, the probability that inexactly 370 of the trials 2 heads are flipped is $\binom{1000}{370}(\frac{3}{8})^{370}(\frac{5}{8})^{630}$.

(d) Approximate the probability of flipping exactly two heads in at least 382 of the trials.
This binomial distribution is approximated by a normal curve with mean $\mu = 1000 \cdot \frac{3}{8} = 375$ and standard deviation $\sigma = \sqrt{1000 \cdot \frac{3}{8} \cdot \frac{5}{8}} = 15.3$. Translating to the standard normal curve give the z-score, $z = \frac{381.5-375}{15.3} = .4245$. The probability that of flipping two heads in at least 382 trials is the area to the right of $z = .4245$ in the standard normal curve, or $1 - .6628 = .3372$.

(8) (a) Consider an absorbing Markov chain with the transition matrix:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} .2 & .4 & .4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & .3 & .5 & .2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the product FR for this matrix?

$$F = \begin{array}{c} 0 \\ 2 \end{array} \begin{bmatrix} \frac{5}{4} & 1 \\ 0 & 2 \end{bmatrix}$$

$$FR = \begin{array}{c} 0 \\ 2 \end{array} \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

(b) Given that we started in state 0, what is the probability that we ended in state 3? $\frac{1}{5}$

(c) Suppose that an initial state from 0,1,2, and 3 is chosen at random, and that we eventually end up in state 1. What is the probability that we started in state 2? Let I_i be the event that our initial state is i, and E_i be the event that our ending state is i. Then by Bayes' Theorem
$P(I_2|E_1) = \frac{P(I_2)P(E_1|I_2)}{P(I_0)P(E_1|I_0)+P(I_1)P(E_1|I_1)+P(I_2)P(E_1|I_2)+P(I_3)P(E_1|I_3)} = \frac{.25(.6)}{.25(.8)+.25(1)+.25(.6)+.25(0)} = \frac{1}{4}$.